# Analysis of the Bullwhip Effect in Supply Chains Using the Transfer Function Method

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## Abstract

An important phenomenon in supply chain management, known as the bullwhip effect, suggests that demand variability increases as one moves up a supply chain. This paper examines the influence of different replenishment policies on the occurrence of the bullwhip effect. The paper demonstrates that certain replenishment policies can in themselves be inducers of the bullwhip effect, while others inherently lower demand variability. The main causes of increase in variability are projections of future demand expectations, which result in over-exaggerated responses to changes in demand. We suggest that, through appropriate selection and use of certain replenishment rules the bullwhip effect can be avoided, subsequently allowing supply chain management costs to be lowered.

**Keywords**: Supply chain management, Bullwhip effect, Replenishment rule, Transfer function, Variability

## Analiza učinka biča v dobavnih verigah z uporabo metodologije prenosne funkcije


### Povzetek

Pojav povečanja variabilnosti naročil, ko se pomikamo navzgor po dobavni verigi, imenujemo učinek biča (angl. Bullwhip effect). Čim višje po dobavni verigi gremo, tem večja bo variabilnost naročil. V pričujočem članku raziskujemo vpliv uporabe različnih sistemov uravnavanja zalog na pojav učinka biča. V članku pokažemo, da so določeni sistemi uravnavanja zalog sami povzročitelji učinka biča, medtem ko uporaba drugih sama po sebi zmanjšuje variabilnost v dobavni verigi. Glavni razlog povečanja variabilnosti so projekcije o prihodnjem povpraševanju, ki se kažejo kot pretiran odziv na spremembe v trenutnem dejanskem povpraševanju. Predlagamo, da se z ustrezno izbiro in uporabo sistema uravnavanja zalog, lahko izognemo učinku biča in s tem posledično zmanjšamo stroške upravljanja dobavne verige.



**Ključne besede**: oskrba dobavne verige, učinek biča, sistem uravnavanja zalog, prenosna funkcija, variabilnost

## 1. Introduction

An important observation in supply chain management was made by Forrester (1961), who illustrated the effect of the variance amplification, called the bullwhip effect, in a series of case studies. The bullwhip effect is a tendency for small changes in end-consumer demand to be amplified as one moves further up the supply chain. Common practical effects of this variance amplification were described in cases of companies Procter&Gamble and Hewlett-Packard, and are presented to students worldwide through the business game “Beer Game” developed on MIT (Sterman, 1989). In 1997 Lee et al. identified five major causes of the bullwhip effect which were all the consequence of the rational behaviour of the supply chain members: use of demand forecasting, batch purchasing, lead times, supply shortages and price fluctuations. For a comprehensive analysis of the bullwhip effect Lee et al. (1997a, b), Metters (1996), Baganha and Cohen (1998) and Chen et al. (2000).

This paper attempts to gauge the impact of demand forecasting and lead times on the bullwhip effect. It has been demonstrated that the replenishment policies used in inventory management combined with demand forecasting can in themselves be the inducers of the bullwhip effect. For order-up-to-level policy Chen et al. (2000) demonstrated that: “if a retailer periodically updates the mean and the variance of demand based on observed customer demand data, then the variance of the orders placed by the retailer will be greater than the variance of demand”. It is known that order-up-to-level policy minimises the inventory and shortage costs (when no fixed costs are considered). However, this is not true if high production switching or fixed ordering costs are incurred, particularly where there is a highly variable order pattern. This has to be taken into careful consideration also because the bullwhip effect is certain to occur when using order-up-to-level policy. This information led us to focus attention on replenishment policies for which demand pattern smoothing may achieve the reduction or even elimination of the bullwhip effect.

The transfer function method used in this paper differs from the more common statistical inventory control approach to explain the occurrence and to quantify the bullwhip effect, used by Lee et al. (1997a, b), Chen et al. (2000), Xu et al. (2001) and Bai (2001). Transfer function analysis of inventory management systems was first done by Simon (1952) by using the Laplace transform. Due to the discrete nature of periodic review replenishment systems, the usage of discrete *Z*-transform was introduced later. In this regard we refer to more recent work from Towill (1999) and, in particular, Dejonckheere et al. (2002a, b). Dejonckheere et al. introduced a new metrics for the bullwhip effect based on the transfer function’s frequency response plot, an issue we will touch on later in this paper.

This paper differs from previous work mainly because it shifts the focus off of the well established and extensively researched order-up-to-level policy and instead looks at replenishment policies that are somewhat different. The basis for our research is a general linear replenishment rule introduced by Bowman (1963), which allows for order inventory smoothing. This rule is then broken down into four simplified rules, among them an order-up-to-level replenishment rule. For each of the rules we will quantify the increase in the variability of orders over the variability of demand.

We begin with a short overview of the transfer function method (Section 2). In Section 3 we continue by constructing a simple supply chain model consisting of a single retailer and a single manufacturer. The starting point is Bowman's replenishment rule, which is the basis for deriving the five analysed replenishment rules. With the use of control engineering methods, we then go on to calculate the transfer functions for all of the analysed replenishment rules. The demand variability amplification is studied through the analysis of the transfer function's frequency response plots in Section 4. The analysis is supplemented with the quantification of the bullwhip effect for given demand patterns. In Section 5 comments are made on the selection of an appropriate replenishment policy with the help of a spreadsheet cost analysis.

## 2. Transfer Function Method

The method used in this paper is transfer function analysis used in control engineering to describe the functioning of a control system that governs the dynamics of a certain process. The connection between the control system and replenishment policy is that with the use of a replenishment policy we correspondingly manage the ordering - replenishment process. The methods encompassed are transfer function derivation, frequency response plots and spectral analysis. The transfer function method is complemented with spreadsheet analysis in order to obtain further insight into system responses, particularly cost analysis. Below are the basic concepts and techniques which constitute a control engineering approach to bullwhip effect analysis:

### 2.1. Transfer Function

By the term "control system" we understand a combination of elements (components of the system) which enable us to control the dynamics of the selected process in certain way. We provide the system with the reference signal (input) that gets translated in the control system and comes out as a control signal (output), thereby controlling the process. In mathematical form we describe the control system's function with its transfer function $G(t)$, which is defined as a ratio of the control system's output $e_2(t)$ and input $e_1(t)$.

$$G(t) = \frac{e_2(t)}{e_1(t)} \tag{1}$$

Since transfer function relates the system's output to its input, knowing the transfer function is enough to precisely describe the function of the control system. Analysis of a particular control system is therefore done through analysis of its transfer function. If the control system is linear, the analysis is simplified with the use of the Laplace transformation and a move from time space into space $s$, the space of the Laplace operator (D'Azzo, Houpis, 1966). Furthermore, if we are dealing with discrete signals, it is necessary to move into $z$ space, through the discrete $Z$ - transform (Houpis, Lamont, 1985). The transfer function is therefore defined as a ratio of the $Z$-transform of the output signal to the $Z$-transform of the input signal and can be written as a ratio of two polynomials with $z$ being a variable:

$$G(z) = \frac{b_0 + b_1 z + b_2 z^2 + \ldots + b_q z^q}{a_0 + a_1 z + a_2 z^2 + \ldots + a_p z^p}, \tag{2}$$

One important property of the $Z$-transform, which we will use extensively, is the translation theorem

$$Z\left[e*(t - pT)\right] = z^{-p} E(z), \tag{3}$$

where $z^{-p}$ is the operator of a time delay in space $z$ and corresponds to a time delay of $p$ time sampling periods $T$ of a discrete signal $e(t)$; $E(z)$ being $Z$-transform of $e(t)$.

The replenishment rules analysed in the paper are all linear and based on periodic review of inventories with a constant review interval, therefore they are all suited for transfer function analysis. We will derive a transfer function for every replenishment rule, where input to the system corresponds to the demand pattern and output refers to the corresponding replenishment or production orders. The derivation of the transfer function through the construction of block diagrams is presented in Section 3.

## 2.2. Frequency Response Plots

The frequency response of the system is a steady state response to the harmonic input signal (sinusoidal) of different frequencies. In dealing with linear systems, the output will also be a sine wave with the same frequency, but the amplitude and the phase angle can change. This amplitude change of the generated sinusoidal output over the sinusoidal input is of particular interest to us. Again we use the transfer function expressed as a ratio of the output over the input signal, but now we are specifically interested in the signal's amplitude change. Since we made a transition into $z$ space when deriving the transfer functions of analysed replenishment, rules we can write the amplitude change or amplitude frequency response as $M_z(\omega T)$. This is done by letting $z = e^{i\omega T}$ in the transfer function $G(z)$ and calculating the modulus of the vector in the complex plane (Houpis, Lamont, 1985):

$$M_z(\omega T) = \left| \frac{E_2(i\omega T)}{E_1(i\omega T)} \right| = \left| G(i\omega T) \right| \tag{4}$$

The frequency response plot depicts the output-input amplitude ratio for sine waves of frequencies $\omega$, ranging from 0 to $\pi$ radians per sampling period $T$.

Since the bullwhip effect can be defined as a variance amplification of orders over demand, the amplitude frequency response plot gives us the magnitude of the bullwhip effect for a sinusoidal demand patterns of frequencies $\omega \in$ [0,. $\pi/T$].

## 2.3. Spectral Analysis

Generally, the input signal in a control system, $e_1(t)$, can be any time series, but we can see it as being composed of sinusoids, $s_i(t)=A_i \sin(\omega_i t)$, of different frequencies, with a particular amplitude $A$ and phase angle, associated with each frequency.

$$e_1(t) = C + s_1 + s_2 + \ldots + s_{(n/2-1)} \tag{5}$$

This can be achieved using a mathematical technique called spectral analysis. The well-known FFT method (Fast Fourier Transform) was used in this paper to perform spectral analysis on real-life demand patterns to obtain demand periodogram. The periodogram represents the relationship between the amplitude, $A$, and frequency, $\omega$, for all possible $n/2$-1 sine waves, where $n$ corresponds to the total number of time periods of demand data (in our case $n$=100). The amplitudes of sine waves depicted in the periodogram are a measure of their relative importance in recomposing the original time pattern. As we have already pointed out, amplitude frequency response $M_z$ gives us the extent of variance amplification for a sinusoidal input signal. Through spectral analysis we are able to break down any demand pattern into a series of sine waves with different frequencies and present it in the form of periodogram. Together, this allows us to determine the variance amplification of every arbitrary demand pattern and, thus, to quantify the bullwhip effect. This approach was previously used by Dejonckheere et al. (2002b) to introduce a new metrics to quantify the bullwhip effect in the supply chain. We use the proposed metrics to calculate the magnitude of the bullwhip effect for generated demand patterns in Section 4.

## 3. Replenishment Policies and Their Transfer Functions

Consider a simple supply chain consisting of a single retailer and a single manufacturer: in each period *t*, the retailer first receives the order from the manufacturer, after which customer demand $D_t$ is observed and filled. Any unfilled demand is backlogged. The retailer observes the new inventory level and forecasts the demand for the next period. Finally, the replenishment order $O_t$ is placed with the manufacturer. There is a fixed replenishment lead time $L$ between the time the order is placed by the retailer and the time it is received by the retailer, such that an order placed at the end of the period $t$ is received at the beginning of the period $t+L$.

### 3.1. Demand Forecasting Policy and Replenishment Rules

Throughout the paper we assume that the retailer is using a common method of simple exponential smoothing to estimate a demand forecast for the next period $\hat{D}_t$, that is:

$$\hat{D}_t = \hat{D}_{t-1} + \alpha(D_t - \hat{D}_{t-1}) \tag{6}$$

Observe that, with the notation used, $D_t$ represents the observed customer demand from the previous period, which we tried to predict by the demand forecast made in the previous period $t$-1, $\hat{D}_{t-1}$. This is possible since we make ordering decisions at the end of the period, after customer demand has already been observed.

In order to explore the replenishment policy induced bullwhip effect we use a general linear replenishment rule introduced by Bowman (1963), which allows for order and inventory smoothing:

$$O_t = \hat{D}_t + (1-\gamma)(O_{t-1} - \hat{D}_t) + \beta(IP_t^T - IP_t), \tag{7}$$

The order quantity $O_t$ is derived based on the demand forecast for the next period $\hat{D}_t$, which is then corrected by the extent of misalignment between the last placed order quantity $O_{t-1}$ in the previous time period and the demand forecast, as well as by the extent of misalignment between the current inventory position $IP_t$ (net stock plus stock on order) and the target inventory position $IP^T{}_t$. This replenishment rule is then broken down into four simpler replenishment rules through the manipulation of the parameters $\beta$ (parameter of inventory position smoothing) and $\gamma$ (parameter of order quantity smoothing)[1]. In Table 1 we give an overview of all five analysed replenishment policies and corresponding replenishment rules.

The first replenishment rule, ($R$, $\hat{D}$), is basically a simple exponential smoothing equation, where order quantity always follows the demand forecast for the next period.

If we rearrange the ($R,\gamma O$) rule equation slightly, we get:

$$O_t = O_{t-1} + \gamma(\hat{D}_t - O_{t-1}), \qquad (8)$$

which is similar in form to the equation for simple exponential smoothing, where order quantity $O$ plays the role of demand forecast $\hat{D}_t$. The rule allows for order quantity smoothing where parameter $\gamma$ has the same role as smoothing constant α in simple exponential smoothing.

Table 1. The five analysed replenishment policies, their rules and corresponding parameter values.

| Replenishment Policy (notation) | Replenishment Rule | Parameter Value $\beta$ | Parameter Value $\gamma$ |
|---|---|---|---|
| $(R, \hat{D})$ | $O_t = \hat{D}_t$ | 0 | 1 |
| $(R,\gamma O)$ | $O_t = \hat{D}_t + (1-\gamma)(O_{t-1} - \hat{D}_t)$ | 0 | $0 \le \gamma \le 1$ |
| $(R,S)$ | $O_t = \hat{D}_t + (IP_t^T - IP_t)$ | 1 | 1 |
| $(R,\beta IP)$ | $O_t = \hat{D}_t + \beta(IP_t^T - IP_t)$ | $0 \le \beta \le 1$ | 1 |
| $(R,\gamma O,\beta IP)$ | $O_t = \hat{D}_t + (1-\gamma)(O_{t-1} - \hat{D}_t) + \beta(IP_t^T - IP_t)$ | $0 \le \beta \le 1$ | $0 \le \gamma \le 1$ |

The third rule we derive from our original Bowman's rule is a well known order-up-to-level policy, ($R,S$) (Silver, Peterson, 1985). The policy is somewhat more complicated then the previous two, due to the introduction of new concepts into the replenishment rule, such as the inventory position, lead time and safety stock. We are particularly interested in changes in inventory position, which goes on to determine the order quantity. In periodic review policies, the time that elapses between two consecutive moments at which we review the stock level is review interval $R$, which is defined in advance and is constant. The time it takes the

[1] Using the notation ($R,\gamma O,\beta IP$) for Bowman's rule, we can say the described replenishment policy is a periodical review policy ($R$) with the review interval $R$, with order quantity smoothing, ($\gamma O$), and inventory position smoothing ($\beta IP$).

manufacturer to fulfil its order is replenishment lead time $L$. For example, if there is no lead time, an order placed at the end of the period $t$ is received and taken into account at the start of the next review period $t+R$. It has to be noted that the key period over which protection is required is of duration $R+L$ instead of just replenishment lead time $L$. In selecting the order-up-to-level $S_t$ at time $t$, we must recognize that, once we have placed an order, no later orders can be received until time $t+R+L$:

$$S_t = \hat{D}_t^{R+L} + SS_t\,, \tag{9}$$

thus $\hat{D}_t^{R+L}$ is forecast demand over $R+L$ periods ($\hat{D}_t^{R+L} = \hat{D}_t(R+L)$). $SS_t$ is a safety stock level ($SS_t = z\hat{\sigma}_t^{R+L} = \hat{\sigma}_t\sqrt{R+L}$), where $\hat{\sigma}_t^{R+L}$ is an estimate of the standard deviation of the probability distribution of forecast demand over key period, $R+L$. To fulfil the linearity condition we have to rewrite the safety stock equation as

$$SS_t = k\hat{D}_t\sqrt{R+L}\,, \tag{10}$$

where standard deviation of the demand is written as a constant part of forecasted demand and $k$ defines a desired service level times the ratio of the standard deviation over the forecast demand ($k\hat{D}_t = z\hat{\sigma}_t$). We made this simplification so that there is only one new parameter, $k$, introduced into the replenishment rule; square root time dependence is therefore still preserved and the linearity condition satisfied.

In order to make the $(R,S)$ rule equations consistent with the notation used in Bowman's rule, we have to set $R$=1, so that the time between the previous and present ordering decision made at $t$-1 and $t$ equals to 1. The time period $R+L$ that determines the order-up-to-level and safety stock level will then be transformed into $1+T_L$, where time $R$=1 corresponds to the review interval and time $T_L$ to the replenishment lead time ($0<T_L<\infty$). It should be noted that this has not affected the generality of our model.

The relevant equations for order-up-to-level $(R,S)$ replenishment policy can finally be written as:

$$S_t = \hat{D}_t(1+T_L) + SS_t \tag{11}$$

$$SS_t = k\hat{D}_t\sqrt{1+T_L} \tag{12}$$

$$IP_t = IP_{t-1} + O_{t-1} - D_t \tag{13}$$

The ($R,\beta IP$) replenishment rule is a variation of the ($R,S$) rule that enables the inventory position smoothing through the loosening of the condition $\beta$=1. In the ($R,S$) rule the misalignment between the current inventory position and the target inventory position was taken into account as a whole, now the correction is partial. We can write the target inventory position as

$$IP_t^T = \hat{D}_t T_L + SS_t \,, \tag{14}$$

where the target inventory position is basically order-up-to-level reduced by the demand forecast for the next period. In ($R,S$) policy we increase the inventory position to the desired level by placing an order so that the current inventory position meets the order-up-to-level. Since order-up-to-level reflects the expected (forecast) demand in the next time period $1+T_L$, we project our future demand expectations over the whole time period $1+T_L$. In ($R,\beta IP$) policy our expectations are projected only over a certain part of a time period $1+T_L$, defined by the inventory smoothing parameter $\beta$. In the extreme case of $\beta$=0 our projections are made only over a time period up to the next ordering decision. In this case ($R,\beta IP$) policy converges to ($R, \hat{D}$) policy.

## 3.2. Transfer Function Calculation

The basis for calculating the transfer function of a particular replenishment policy is its replenishment rule. We represent the dynamics of the system through the construction of the causal-loop diagram. From here we construct a block diagram (Figure 1). The input in the block diagram of each of our replenishment policies is the demand signal, which is the only independent variable in the inventory replenishment system. The corresponding output on the opposite side of the diagram is the order quantity.

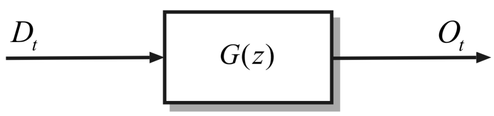



Figure 1. Block diagram for the replenishment rule

Let us begin by deriving the transfer function of simple exponential smoothing. The forecast demand is basically a mix of actual demand observed and past forecasts. To introduce past forecasts in the demand forecast we develop a feedback loop within the block diagram. Note the delay operator $z^{-1}$ we have introduced by the $Z$-transform translation theorem (Eq. (3)). By applying the rules for block diagram reduction we obtain the following transfer function as a relationship between the demand forecast and the actual observed demand

$$G(z)=\frac{\hat{D}}{D}=\frac{\alpha z}{z-(1-\alpha)} \tag{15}$$

We follow the same steps in deriving the transfer functions of the five replenishment policies. We begin by constructing the block diagram for the simplest ($R, \hat{D}$) policy, which we then gradually extend to the most comprehensive diagram, for ($R,\gamma O,\beta IP$) policy. The block diagram of simple exponential smoothing is always incorporated into each of the replenishment rule block diagrams that are given in Figure 2. The ($R,\gamma O,\beta IP$) policy block diagram basically consist of the three major components: exponential smoothing feedback loop, order smoothing feedback loop and inventory position smoothing feedback loop. Here we should note that the actual observed demand signal is directly taken into account only in the last three replenishment policies (into the inventory feedback loop), the consequences of this will be discussed in greater detail later. Transfer functions of the analyzed replenishment rules are expressed as a ratio of orders and observed demand. In Table 2, we present the transfer functions of all five analysed replenishment policies.

Table 2. The transfer functions of the five analysed replenishment policies.

| Replenishment Policy (notation) | Transfer Function |
|---|---|
| $(R,\hat{D})$ | $\frac{O}{D}=\frac{\alpha z}{z-(1-\alpha)}$ |
| $(R,\gamma O)$ | $\frac{O}{D}=\frac{\alpha\gamma z^2}{[z-(1-\alpha)][z-(1-\gamma)]}$ |
| $(R,S)$ | $\frac{O}{D}=\frac{\left((T_L+1)+k\sqrt{T_L+1}\right)\alpha(z-1)}{z-(1-\alpha)}+1$ |
| $(R,\beta IP)$ | $\frac{O}{D}=\frac{\left(1+\beta\left(T_L+k\sqrt{T_L+1}\right)\right)\alpha z(z-1)+\beta z[z-(1-\alpha)]}{[z-(1-\alpha)][z-(1-\beta)]}$ |
| $(R,\gamma O,\beta IP)$ | $\frac{O}{D}=\frac{\left(\gamma+\beta\left(T_L+k\sqrt{T_L+1}\right)\right)\alpha z^2(z-1)+\beta z^2[z-(1-\alpha)]}{[z-(1-\alpha)][z(z-1)-(1-\gamma)(z-1)+\beta z]}$ |

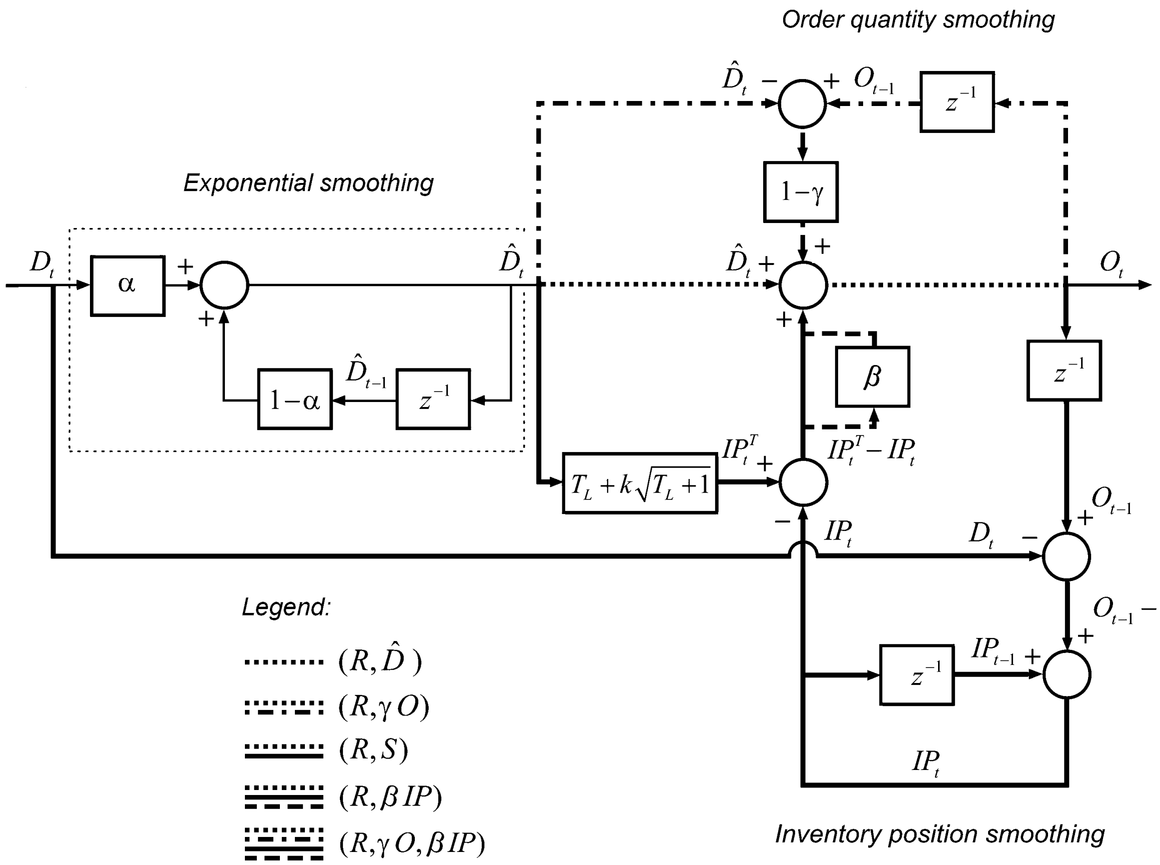



Figure 2. Block diagram for the five replenishment policies analysed.

# 4. Transfer Function Analysis of the Replenishment Policies

By analysing the transfer functions of the replenishment policies we want to test whether the input or demand signal amplifies when it goes through the inventory management system (using certain replenishment policy) and comes out as an output or order quantity signal. If amplification is noted there is increase in variance of the signal and therefore the bullwhip effect occurs.

## 4.1. Frequency Response Analysis

As previously stated, the frequency response plot gives us the output to input amplitude change $A_2/A_1$ for a given input sinusoidal signal $s_1(t)$ of frequency $\omega$. Since variance of a sinusoidal signal can be expressed as the square of the amplitude $A$ divided by two ($var\ s(t)=A^2/2$), it can be easily shown that the output to input amplitude ratio $M_z(\omega)$ is exactly the same as the ratio of the standard deviation of the order pattern over the standard deviation of the demand pattern:

$$M_z(\omega) = \frac{A_2}{A_1} = \sqrt{\frac{\operatorname{var} s_2}{\operatorname{var} s_1}} = \frac{\sigma(s_2)}{\sigma(s_1)}, \quad (16)$$

where ratio of standard deviations is an accepted metrics for bullwhip effect quantification. We have already used the review interval presumption of $R$=1, which corresponds to the sampling period $T$ in Eq. (4) and frequency interval from 0 to π radians in frequency response plots.

The frequency response plot therefore shows whether the sinusoidal demand patterns of a certain frequency were amplified or weakened; whether their variability had increased or decreased. If the frequency response lies above 1 for a given frequency $\omega$, it means that the demand signal is amplified and bullwhip effect exists. Since the frequency response plot depicts all possible frequencies, we can give a general observation of possible demand variability increase, hence the bullwhip effect occurrence. In Figure 3 to Figure 6, we present the frequency response plots for the five replenishment policies analysed in the paper. We present some vital observations from the frequency response plots, which are then supplemented with further comments in Section 5.

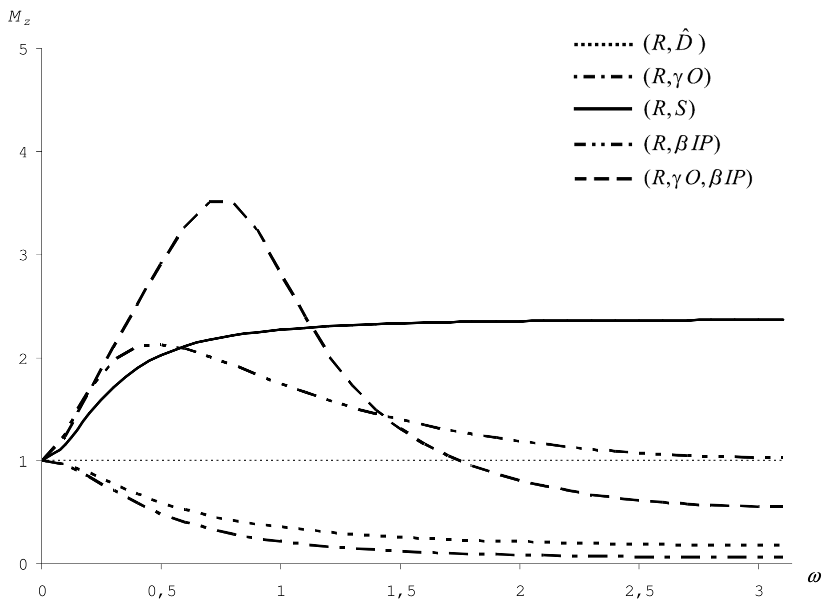



Figure 3. The frequency response plot for the replenishment rules analysed ($\alpha$= 0.3, $\gamma$=0.5, $\beta$=0.5, $T_L$=2, $k$=0.5).

The frequency response plot for all five analysed replenishment rules is given in Figure 3. We see that, for $(R, \hat{D})$ and $(R, \gamma O)$ replenishment policies, a distinct variance reduction is present, more so in the case of $(R, \gamma O)$. The frequency response plot is below 1 throughout, which means the bullwhip effect will not occur for every possible demand pattern. Variance reduction is greater in the high frequency region, where the variability can decrease by half or more. We should note that, with the frequency value $\omega=0$, the demand pattern is constant and there is no increase in variability of orders over demand; the order pattern is therefore expected to equal the demand, hence there is no bullwhip effect. This is also true in the case of the remaining three replenishment policies.

On the contrary there is an overshoot in the frequency response plot of order-up-to-level - $(R,S)$ policy for all possible frequencies. In line with our previous statements, it is clear that, in this case, the bullwhip is guaranteed no matter what the demand pattern will be. This is consistent with the before mentioned findings of Chen et al. (2000). The frequency response plots for the remaining two policies, $(R,\beta IP)$ and $(R, \gamma O, \beta IP)$, are more varied. For low frequency region, there is a notable overshoot, but in the region of high frequencies, a reduction in variance can be achieved through the proper selection of rule's parameters. This is especially the case with $(R, \gamma O, \beta IP)$ policy, where we notice a high peak in a low frequency region followed by a strong drop towards higher frequencies.

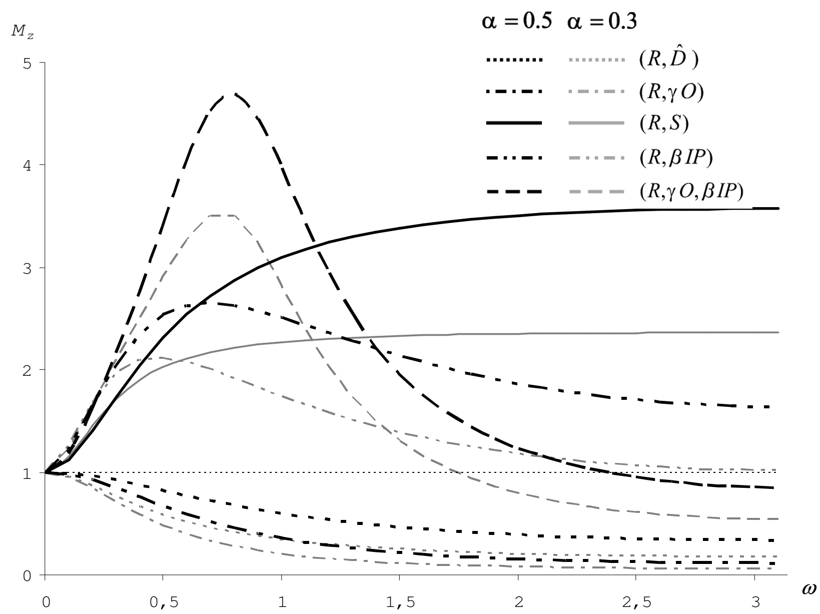



Figure 4. The frequency response plot for the replenishment rules analysed, changing the exponential smoothing parameter $\alpha$ ($\alpha$= {0.3, 0.5}, $\gamma$=0.5, $\beta$=0.5, $T_L$=2, $k$=0.5).

The demand forecast exponential smoothing parameter $\alpha$ has a major impact on the bullwhip effect (Figure 4). Choosing a small $\alpha$ value smoothes out the unstable demand pattern; this is in turn reflected in lower variability of orders, meaning a reduction in possible bullwhip effect is expected. We can confirm this by considering the frequency response plots of all five replenishment policies. However, we see that the bullwhip effect can not be totally eliminated in the case of (*R,S*) policy. We will explain the reasons for this later. If we employ a responsive policy by choosing a large $\alpha$, the variability of orders can be up to 5 times higher than the variability of demand. As we have already mentioned, we are faced with a dilemma between being responsive (thus following the demand changes very closely) and avoiding the bullwhip effect. The selection of an exponential smoothing parameter for ($R,\beta IP$) and ($R,\gamma O,\beta IP$) policies can make a difference between experiencing and avoiding the bullwhip effect.

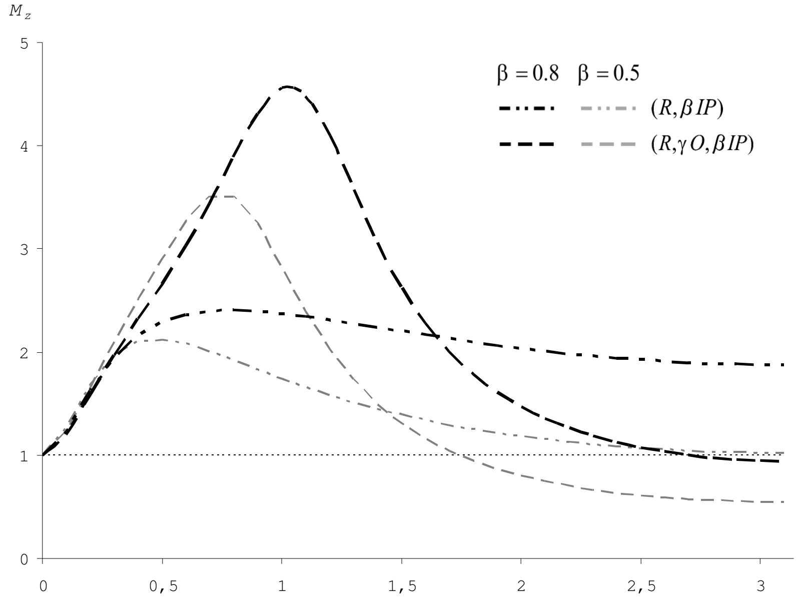



Figure 5. The frequency response plot for the replenishment rules analysed, changing the inventory position smoothing parameter $\beta$ ($\alpha$=0.3, $\gamma$=0.5, $\beta$={0.5, 0.8}, $T_L$=2, $k$=0.5).

The original idea, by using Bowman's replenishment rule and its derivatives, was trying to find a low or even no bullwhip alternative to order-up-to-level policy. This was to be done through order quantity smoothing and inventory position smoothing. From frequency response plots given in Figure 5 and Figure 6 we see that inventory position smoothing in ($R,\beta IP$) and ($R,\gamma O,\beta IP$) policies have a certain effect on bullwhip reduction, although the influence of inventory position smoothing parameter $\beta$ is not as great as in the case of parameter $\alpha$. The higher the $\beta$ the smaller the variance reduction. There is also a noticeable shift of peak in variance amplification towards higher frequencies for larger $\beta$ values in $(R,\gamma O,\beta IP)$ policy. Order quantity smoothing in $(R,\gamma O)$ and $(R,\gamma O,\beta IP)$ policies generally also insures additional bullwhip effect reduction, except in the low frequency region of the $(R,\gamma O,\beta IP)$ frequency response plot. We believe this is due to the occurring oscillations, particularly if small $\gamma$ values are chosen, an issue that can be addressed through transfer function's poles analysis. The transfer function of the $(R,\gamma O,\beta IP)$ replenishment rule has generally a pair of complex poles, except for values of $\beta \rightarrow 0$ and $\gamma \rightarrow 1$.

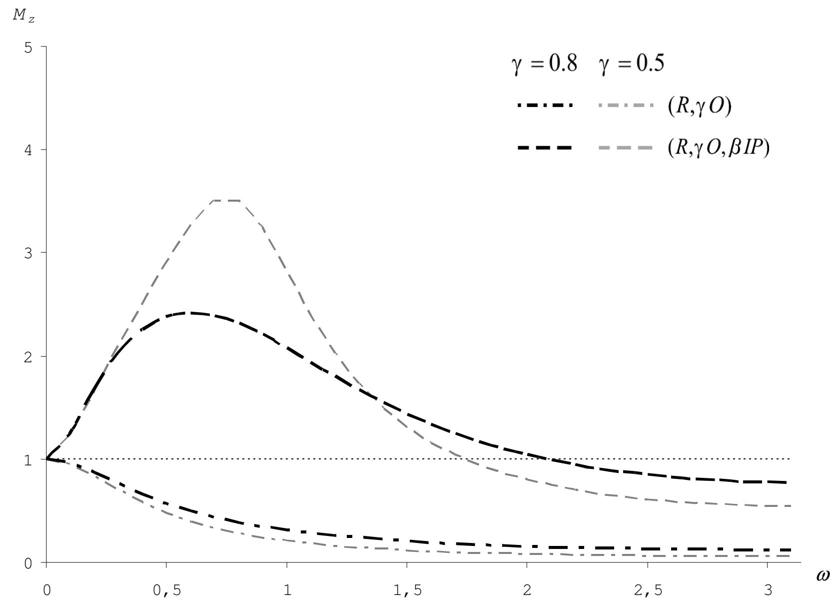



Figure 6. The frequency response plot for the replenishment rules analysed, changing the order quantity smoothing parameter $\gamma$ ($\alpha$=0.3, $\gamma$={0.5, 0.8}, $\beta$=0.5, $T_L$=2, $k$=0.5).

There are still two parameters in our replenishment rule model that are of interest to us: lead time $T_L$ and the safety stock parameter, $k$. It is well known that non-zero lead times contribute to variance amplification (Lee et al., 1997a). Following on from this, a rise in the height of the frequency response plots of all three: $(R,S)$, $(R,\beta IP)$ and $(R,\gamma O,\beta IP)$ policies is anticipated. However, there is no change in the shape of the frequency response curves. Therefore the lengthening of the lead time leads to an almost proportional increase in the bullwhip effect. We can draw the same conclusions the same for the effect of changing safety stock levels through the manipulation of parameter $k$. The higher the safety stock levels, the higher the bullwhip effect.

## 4.2. Bullwhip Effect Quantification for Generated Demand Patterns

To quantify the variance amplification of any given demand pattern we used the metrics proposed by Dejonckheere et al. (2002b). Each of the demand patterns (input) was decomposed into the sum of $n/2$-1 sine waves (Eq. (5)). Because the variance of the $i$th sine wave is equal to $A_i^2/2$ and the covariance between two sine waves with different frequencies is always zero, the variance of the input signal can be written as:

$$\operatorname{var} e_1 = \tfrac{1}{2}\sum_{i=1}^{N/2-1} A_{1,i}^2 \tag{17}$$

We then used the frequency response plot (Eq. (16)) to find the corresponding amplitudes of the order pattern (output) and write the variance of the output signal as:

$$\operatorname{var} e_2 = \tfrac{1}{2}\sum_{i=1}^{N/2-1} A_{1,i}^2 M_z^2(\omega_1) \tag{18}$$

As in Eq. (16) the ratio of the two above equations gives us the measure of the Bullwhip effect, $U$:

$$U = \sqrt{\frac{\operatorname{var} e_2}{\operatorname{var} e_1}} = \sqrt{\frac{\sum_{i=1}^{N/2-1} A_{1,i}^2 M_z^2(\omega_i)}{\sum_{i=1}^{N/2-1} A_{1,i}^2}} \tag{19}$$

We applied the described metrics to ten generated demand patterns for all five replenishment policies analysed. The demand patterns were generated using the following equation (where demand pattern is a sum of constant demand (parameter $a$), positive or negative trend (parameter $b$), seasonal component represented by sinusoidal function (parameters E and $\nu$) and random component (parameter $\sigma$)):

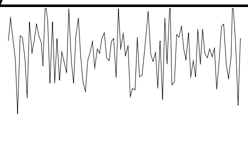
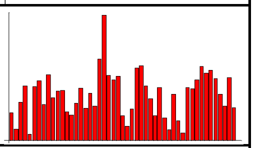
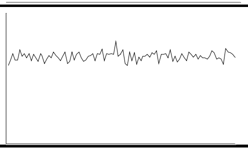
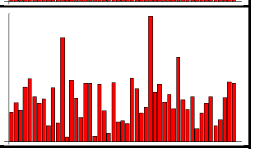
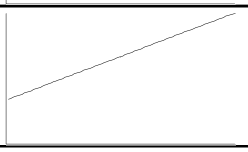
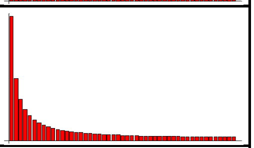
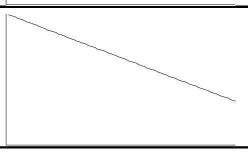
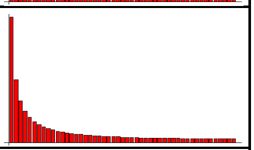
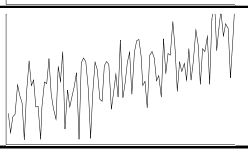
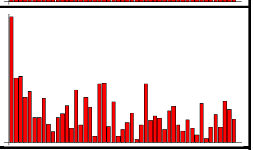
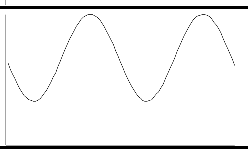
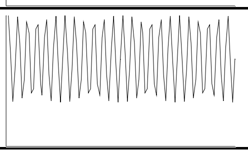
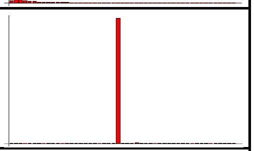
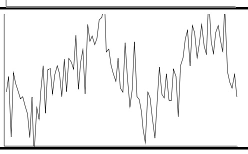
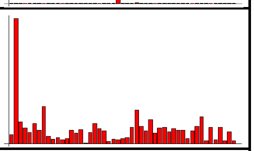
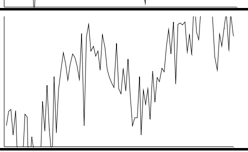
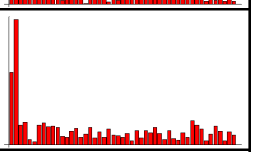
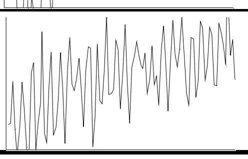
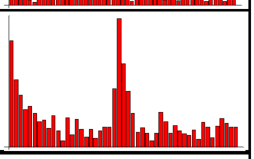
$$D_t = a + bt + F_t + \varepsilon_t = a + bt + \mathrm{E}\sin(2\pi\nu \cdot t) + N(0,\sigma) \tag{20}$$

Note the measurement of the bullwhip effect has no requirements choosing certain demand pattern; the metrics allow for bullwhip effect quantification for any given demand pattern. We chose the above equation for simple generation of some characteristic and, also in some cases, quite realistic demand patterns.

We present the results in Table 3, where the measured bullwhip effect is shown together with the parameters of generated demand patterns, plots of time demand patterns and corresponding periodograms. We found that the results entirely corroborate the observations obtained from the frequency response plots' analysis made earlier.

To verify these results we run the same demand patterns through the spreadsheet simulation and measure the bullwhip effect. The average deviation from results obtained with the transfer function methodology is in most cases small enough to confirm the appropriateness of the transfer function-based metrics.

Table 3. Bullwhip effect quantification for the ten generated demand patterns.

| **Demand parameters** | | | **Bullwip effect for replenishment policies** $\alpha$=0.3, $\gamma$=0.5, $\beta$=0.5, $T_L$=2, $k$= 0.5 | | | | | | **Demand time diagram** | **Demand periodogram** |
|---|---|---|---|---|---|---|---|---|---|---|
| **N** | | | | $(R,D)$ | $(R,\gamma O)$ | $(R,S)$ | $(R,\beta IP)$ | $(R,\gamma O,\beta IP)$ | | |
| 1 | a | 100 | *U G* | 0,349 | 0,259 | 2,270 | 1,289 | 1,477 |  |  |
| | b | | *U Sim* | 0,354 | 0,265 | 2,268 | 1,466 | 1,740 | | |
| | σ | 30 | *error* | 1,4% | 2,3% | 0,1% | 12,1% | 15,1% | | |
| | E | | | | | | | | | |
| | ν | | | | | | | | | |
| 2 | a | 100 | *U G* | 0,390 | 0,313 | 2,236 | 1,284 | 1,529 |  |  |
| | b | | *U Sim* | 0,389 | 0,308 | 2,240 | 1,491 | 1,890 | | |
| | σ | 5 | *error* | 0,3% | 1,6% | 0,2% | 13,9% | 19,1% | | |
| | E | | | | | | | | | |
| | ν | | | | | | | | | |
| 3 | a | 100 | *U G* | 0,929 | 0,909 | 1,212 | 1,265 | 1,365 |  |  |
| | b | 1 | *U Sim* | 0,919 | 0,904 | 1,317 | 1,354 | 1,472 | | |
| | σ | | *error* | 1,1% | 0,6% | 8,0% | 6,6% | 7,3% | | |
| | E | | | | | | | | | |
| | ν | | | | | | | | | |
| 4 | a | 100 | *U G* | 0,929 | 0,909 | 1,212 | 1,265 | 1,365 |  |  |
| | b | -1 | *U Sim* | 0,919 | 0,904 | 1,317 | 1,354 | 1,472 | | |
| | σ | | *error* | 1,1% | 0,6% | 8,0% | 6,6% | 7,3% | | |
| | E | | | | | | | | | |
| | ν | | | | | | | | | |
| 5 | a | 100 | *U G* | 0,682 | 0,640 | 1,845 | 1,330 | 1,578 |  |  |
| | b | 1 | *U Sim* | 0,659 | 0,623 | 1,918 | 1,462 | 1,742 | | |
| | σ | 30 | *error* | 3,5% | 2,7% | 3,8% | 9,0% | 9,4% | | |
| | E | | | | | | | | | |
| | ν | | | | | | | | | |
| 6 | a | 100 | *U G* | 0,945 | 0,932 | 1,251 | 1,300 | 1,260 |  |  |
| | b | | *U Sim* | 0,943 | 0,929 | 1,233 | 1,376 | 1,334 | | |
| | σ | | *error* | 0,2% | 0,3% | 1,5% | 5,5% | 5,5% | | |
| | E | 50 | | | | | | | | |
| | ν | 2 | | | | | | | | |
| 7 | a | 100 | *U G* | 0,254 | 0,117 | 2,331 | 1,228 | 1,129 |  |  |
| | b | | *U Sim* | 0,253 | 0,116 | 2,331 | 1,387 | 1,294 | | |
| | σ | | *error* | 0,4% | 0,9% | 0,0% | 11,5% | 12,8% | | |
| | E | 50 | | | | | | | | |
| | ν | 24 | | | | | | | | |
| 8 | a | 100 | *U G* | 0,779 | 0,755 | 1,713 | 1,312 | 1,376 |  |  |
| | b | | *U Sim* | 0,779 | 0,753 | 1,692 | 1,434 | 1,515 | | |
| | σ | 30 | *error* | 0,0% | 0,3% | 1,2% | 8,5% | 9,2% | | |
| | E | 50 | | | | | | | | |
| | ν | 2 | | | | | | | | |
| 9 | a | 100 | *U G* | 0,838 | 0,820 | 1,547 | 1,309 | 1,411 |  |  |
| | b | 1 | *U Sim* | 0,839 | 0,819 | 1,551 | 1,391 | 1,515 | | |
| | σ | 30 | *error* | 0,1% | 0,1% | 0,3% | 5,9% | 6,9% | | |
| | E | 50 | | | | | | | | |
| | ν | 2 | | | | | | | | |
| 10 | a | 100 | *U G* | 0,582 | 0,538 | 2,020 | 1,284 | 1,352 |  |  |
| | b | 1 | *U Sim* | 0,580 | 0,539 | 2,035 | 1,422 | 1,513 | | |
| | σ | 30 | *error* | 0,3% | 0,2% | 0,7% | 9,7% | 10,6% | | |
| | E | 50 | | | | | | | | |
| | ν | 24 | | | | | | | | |
| **Average** | | | *error* | **0,8%** | **0,9%** | **2,4%** | **8,9%** | **10,3%** | | |

## 5. Insights and Comments on Replenishment Policy Selection

### 5.1. Replenishment Policy as a Generator of the Bullwhip Effect

Through transfer function analysis we showed that there is no bullwhip effect if we use ($R, \hat{D}$) or ($R,\gamma O$) replenishment policy. On the contrary, they both tend to lower the amplification of the orders over the demand. Order quantity in ($R, \hat{D}$) policy equals the forecast demand for the next period; this means that the policy response to the change in actual demand will always be smaller because of the demand forecast smoothing. With ($R,\gamma O$) policy we have another contribution to bullwhip reduction through order quantity smoothing. If we are experiencing a rise in demand, a smaller rise in forecast demand, as in ($R, \hat{D}$) policy, follows. There is also a negative contribution due to the misalignment between the last placed order quantity and the demand forecast, which additionally lowers the change in order quantity. Consequently, the variability of orders is even lower.

For ($R,S$), ($R,\beta IP$) and ($R,\gamma O,\beta IP$) policies, we showed that we can expect the bullwhip effect to occur. This is particularly true for the order-up-to-level ($R,S$) policy. If we want to explore the causes for variance amplification we must take a closer look at the dynamics of the replenishment systems analysed. The original Bowman's ($R,\gamma O,\beta IP$) rule consists of three parts. The first two - demand forecast and order quantity smoothing - are clearly not the generators of increased variance, as was demonstrated in the analysis of ($R, \hat{D}$) and ($R,\gamma O$) policies. So let us look closer at the third part: inventory position smoothing. When we place a new order, order-up-to-level is corrected based on forecast demand. Order quantity is now determined by adding the extent of misalignment between the current inventory position and the target inventory position. We have already said the target inventory position reflects our future demand expectations over a lead time. The longer the lead time, further into the future our expectations are projected. Let us say that we observe a rise in demand. Accordingly, we forecast the future demand using simple exponential smoothing. We know that this smoothes the demand pattern, but the forecast will go slightly up nonetheless. This rise in forecast will not be considered only in the next period, but will be projected over the whole of the lead time through the correction of the target inventory position. The tendency of rising demand will be considered over a proportionally longer time period (the review interval plus the production delay). In the case of long lead times, even minor changes in end-consumer demand can result in a much higher target inventory position. The extent of misalignment between the current inventory position and the target inventory position will be greater and consequently the order quantity will be proportionally higher. If a drop in demand occurs in the next period, our reaction will again be exaggerated and will show in the excessive decrease in order quantity. This, of course, leads to high variance amplification - the bullwhip effect.

This was the reason we introduced inventory position smoothing. With the choice of low inventory position smoothing parameter $\beta$ values, we do not take into account the misalignment between the current inventory position and the target inventory position to the full extent. As a result we effectively shorten the time over which we project our expectations about future demand. Changing expectations will be therefore reflected over a shorter period of time, which will result in bullwhip reduction. We can confirm this by looking at the frequency response plots for ($R,\beta IP$) and ($R,\gamma O,\beta IP$) policies in Figure 5. Choosing low $\beta$ value leads to bullwhip effect elimination. In the extreme case of $\beta$=0, the expectations are projected only over the review interval, which means the current demand forecast will be relevant only until the time of the next ordering decision.

The ($R,\gamma O,\beta IP$) replenishment policy exhibits very interesting behaviour. It could be expected that applying the possibility of order quantity smoothing to the ($R,\beta IP$) policy would result in further bullwhip effect reduction, but this is not the case. This is due to oscillations that appear in the order pattern and contribute to increased variability. More complicated replenishment rule or the combination of two means of smoothing of orders does not guarantee reduction in variability; on the contrary, using ($R,\gamma O,\beta IP$) policy leads to higher bullwhip effect.

It was shown that an increase in safety stock levels leads to a greater bullwhip effect. This is due to the fact that, in given periods, we determine the safety stock level according to the demand forecast and the length of the lead time. There is direct link between the influence of the safety stock levels and length of the lead time on the bullwhip effect. We can say the safety stock level actually increases the lead time, which now represents not only the actual lead time but also an added safety lead time. As a consequence the projection of future demand lengthens for the extent of the safety lead time and this results in higher variability.

### 5.2. Cost Analysis and Replenishment Policy Selection

It is known that order-up-to-level policy (remember our notation: ($R,S$) policy) is optimal in the sense that it minimises the inventory management costs (holding and shortage costs) when no fixed costs are incurred. However, we showed that, using ($R,S$) policy always leads to the bullwhip effect; in fact its magnitude is the greatest experienced among the five policies we analysed. At first glance the above statements are contradictory given that we know that increase in supply chain variability leads to inefficiencies and high costs. To dispel this seeming contradiction we complement the transfer function analysis with a simple spreadsheet cost analysis.

We create two separate cases: in the first, only variable inventory holding and shortage costs are accounted for; while in the second case, we add fixed ordering or production switching costs. Calculation of inventory management costs was done on 20 randomly generated demand patterns with the same mean demand and standard deviation. To reduce the effect of the starting conditions we first ran the simulation for a starting 100 time periods and then measured the magnitude of the bullwhip effect and costs on the following 100 time periods.

In Figure 7 we confirm that in the first case, in which only variable costs where included, ($R,S$) policy has the lowest total costs. ($R,\beta IP$) and ($R,\gamma O,\beta IP$) policies have slightly higher costs, while the costs for ($R,\hat{D}$) and ($R,\gamma O$) policies are the highest. The opposite is true for the bullwhip effect, which is higher for policies with lower total costs. When fixed costs are also included, ($R,S$) policy ceases to be optimal (Figure 8). We can see a relative drop in total costs for ($R,\hat{D}$) and ($R,\gamma O$) policies. Also, ($R,\beta IP$) policy is in both cases able to outperform ($R,\gamma O,\beta IP$) policy.

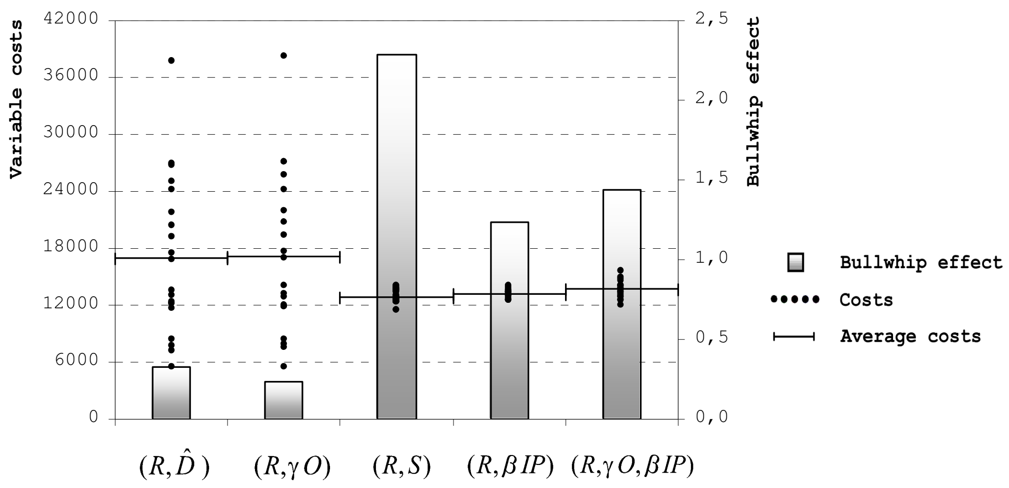



Figure 7. The bullwhip effect and variable inventory costs for the five analysed replenishment policies.

Inventory management cost analysis clearly demonstrates that the choice of an appropriate replenishment rule is a compromise between the reduction of bullwhip effect, on one hand, and the reduction of supply chain costs on the other (when only variable inventory management costs are considered). Based on this the recommendation for choosing the appropriate replenishment rule depends on whether the benefits of the bullwhip effect reduction outweigh the higher inventory management costs. In companies where high variability of orders results in high costs (either due to high ordering or production switching costs), the reduction in bullwhip effect can lead to significant cost savings. Such companies should consider implementing a replenishment policy such as $(R, \hat{D})$ or $(R,\gamma O)$ policy. The company that wants to be highly responsive and for which an increase in variability does not incur high costs, would best use the order-up-to-level policy. The selection of $(R,\beta IP)$ policy represents the middle way and can in certain conditions prove to be optimal. $(R,\gamma O,\beta IP)$ policy is always outperformed by $(R,\beta IP)$ policy.

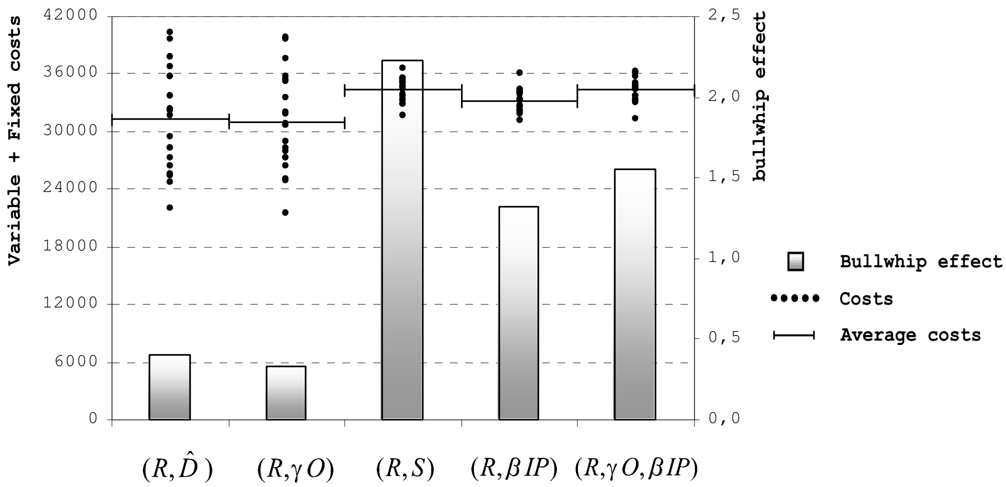



Figure 8. The bullwhip effect and variable plus fixed inventory costs for the five analysed replenishment policies.

When fixed costs were also included in the analysis it turned out that replenishment rules that experience lower bullwhip effect have a clear advantage.

Finally, for $(R, \hat{D})$ and $(R,\gamma O)$ policies, we observe a considerable spread between the calculated inventory related costs. This is due to the simplicity of the two policies, where there is a lack of inventory position tracking. Since inventory position is not reviewed and therefore cannot be corrected, the result can be very high costs if the changes in demand are unfavourable and vice-versa. Even though both policies can exhibit lowest inventory management costs, the wide spread results in volatility that can outweigh the benefits of having lower average costs.

## 6. Conclusions

The main research goal in this paper was to establish the influence of using different replenishment policies on the occurrence and extent of the bullwhip effect. A company's decision to use a certain replenishment policy is a rational decision that should be taken with the aim of improving inventory management and assuring an adequate service level in mind. We have shown that, in some cases, an otherwise "optimal" selection causes the bullwhip effect and thus turns out to be ineffective and inappropriate.

We demonstrated that demand forecasting with simple exponential smoothing, which was an integral part of all five analysed replenishment policies, effectively lowers the variability of orders over demand and thus decreases the probability of the bullwhip effect occurring. However, we also confirmed that some replenishment rules can in themselves be the inducers of the bullwhip effect. This is particularly the case in order-up-to-level ($R,S$) policy, where bullwhip effect is inevitable for any demand pattern. The main generator of increase in variability is future demand projections which result in an over-exaggerated response to changes in demand. With ($R,\beta IP$) and ($R,\gamma O,\beta IP$) replenishment policy, the extent of the bullwhip effect depends on the nature of the demand pattern. The reduction of the bullwhip effect is achieved through the fact that the misalignment between current inventory position and target inventory position is not taken into account to its full extent. On the other hand we showed that some of the replenishment rules, ($R, \hat{D}$) and ($R,\gamma O$) policy, are inherently bullwhip effect "free".

Inventory management cost analysis clearly showed that the choice of an appropriate replenishment rule is a compromise between the reduction of the bullwhip effect, on one hand, and the reduction of supply chain costs on the other (when only variable inventory management costs are considered). When fixed costs were also incorporated into the analysis, it turned out that replenishment rules that experience lower bullwhip effect have a clear advantage. This highlights the importance of choosing the replenishment rule that is best suited to the cost structure of the companies' inventory management system.

Finally, we have to stress again that actual observed demand is not necessarily the end-consumer demand. Subsequently, the analysis of the bullwhip effect presented in this paper is in this view very general and its results can be applied to any of the supply chain links.

The transfer function method used in the paper allowed us to gradually build our replenishment policy analysis by deriving its transfer functions to analysis based on frequency response plots, which finally led us to bullwhip effect quantification. The main advantage in using this method lies in its intuitive results, which are obtained with the graphical illustration of the bullwhip effect through the frequency response plots. This cannot be achieved to such an extent with the more common statistical methods and simulations.

However, there are some limitations to the method used. The choice of the replenishment policies is limited since they have to be inherently periodic review policies and have to satisfy the linearity condition.

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